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An Application of Optimal Control to the Economics of Recycling*

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Abstract. A city's landfill is an exhaustible resource; recycling is a backstop method of waste disposal. We formulate an optimal control model that maximizes the total utility of a representative consumer by choosing the appropriate levels of the two disposal techniques. The model is simple enough that some general conclusions can be drawn, and yet specific solutions will require some computations that show a few of the possibilities in optimal control problems. Among the primary results is that once recycling begins, it will increase. It is also likely that the level of recycling will assume the values at the endpoints of its domain as well as the more standard values in the interior.

Key words. optimal control, recycling, municipal waste management

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Since the 1960s, economists have been using techniques of optimal control to solve both theoretical and applied optimization problems. Seierstad and Sydsæter's book [16] contains many examples of such problems. Optimal control techniques have been used in the exhaustible resources literature, recently in considering the problems of (solid) waste management. Traditionally, landfilling has been the principal method of municipal waste disposal. But since landfill capacity is an exhaustible resource, municipal recycling (the recycling of household waste done by (or for) a municipality) has become increasingly important. Most states in the United States now require some recycling by municipalities [8]; such legislation is often motivated in part by the desire to conserve landfill space. Recycling, however, is a more expensive waste disposal method than landfilling (even allowing for revenue it might generate for the municipality).

Thus, the municipality's waste management problem can be thought of in the context of the exhaustible resources literature as a problem of optimal use of an exhaustible resource (landfill capacity) when a backstop waste disposal technology (recycling) exists. A "backstop technology" is characterized as one that is more expensive but has unlimited capacity. This paper's results are restricted to municipalities with

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only these two methods of waste disposal—a simplifying assumption since, of course, other backstop waste disposal technologies (e.g., incineration) exist.[1]

Similar landfilling/recycling optimal control problems are found in Huhtala [11] and Highfill and McAsey [9, 10]. By considering the net benefits of waste disposal services, Huhtala [11] found conditions for an optimal switch from an old landfill to a new one. The present paper is an extension of Huhtala [11] and Highfill and McAsey [9] in that it allows for income growth over time rather than assuming income to be constant. Highfill and McAsey [10] allowed for income growth but relied heavily on specific functional forms (for utility and costs); the present paper does not. Another difference between the present paper and [10] is that the latter supposed that consumers derive utility from recycling per se, while the present paper supposes that consumers value recycling only indirectly as a method of waste disposal. Other approaches to municipal recycling are found in the literature, e.g., Huhtala [12], Calcott and Walls [1], Fullerton and Kinnaman [6], Fullerton and Wolverton [7], and Ley, Macauley, and Salant [13]. The empirical literature on municipal recycling is growing quickly. Notable recent contributions are Craighill and Powell [2], Tiller, Jakus, and Park [18], Sterner and Bartelings [17], Nestor and Podolsky [14], and Tucker [19]. There is an extensive literature on backstop technologies in a context other than waste management; see Dasgupta and Heal [3], Prell [15], and Endress and Roumasset [4].

The primary result of the paper is the prediction that recycling, once begun, will always increase in a community whose income is increasing. This result holds even though recycling is not an argument of the utility function and the per unit cost of recycling increases with the amount of recycling (and the total recycling cost function is convex). The only exception to the result is trivial: some municipalities have landfills sufficiently large so that they never recycle. By using comparative dynamics, it is also seen that smaller initial landfill capacities imply greater reliance on recycling. The main message for classroom use is that although the necessary conditions from the optimal control problem are certainly important in understanding the optimal path, the endpoint conditions are equally important, and, in general, an optimal path requires an understanding of both the necessary and endpoint conditions. This material is appropriate for a two-day unit in a natural resources course, providing an example of an exhaustible resource/backstop problem in waste disposal technologies. It might also fit in an undergraduate course in modeling, optimization, applied mathematics, or mathematical economics, or even a course in dynamic optimization for graduate students in economics. Several numerical examples are included and two sets of exercises are provided.

1. The Model. A municipality (city, county, etc.) is composed of a number of individuals. The decision making of the municipality is modeled by considering a "representative consumer" who might be thought of as the typical or average resident of the municipality. Let $c(t)$ be the rate of consumption of the aggregate good (i.e., food, shelter, clothing, recreation, etc.) by the representative consumer. All consumption is assumed to become waste which must be disposed of by either landfilling or recycling: $c(t) = x(t) + z(t)$, where $x(t)$ denotes the rate at which waste is recycled and $z(t)$ the rate at which waste is deposited in a landfill.[2]

[1] In 1997 the United States landfilled 55% of all waste; 27% was recycled or composted; and 17% was incinerated. But waste disposal methods vary considerably by region. In Illinois, for example, in 1998, 69% of waste was landfilled, 28% recycled, and 3% incinerated.

[2] The proportion of consumption that becomes waste could be set at something other than 100%. For example, $\alpha \cdot c(t) = x(t) + z(t)$, $0 < \alpha < 1$. The analysis would be essentially unchanged as long as α is exogenous.

Landfill capacity is considered to be an exhaustible resource. Specifically, let s_0 be the initial landfill capacity, $s(t) \geq 0$ be the amount of landfill available at time t, and T be the planning horizon. (Typical planning horizons in counties are 5–15 years.) Landfill use is thus constrained by

$$s(t) = s_0 - \int_0^t z(\tau)\, d\tau.$$

It will be convenient in the optimal control problem to write this constraint as a differential equation: $ds/dt = -z(t)$ with the initial condition $s(0) = s_0$.

Recycling is a more expensive waste disposal method than landfilling; any revenue it may generate is small in comparison with other expenses. Thus, recycling is a backstop waste disposal technology. Denote the cost per unit of recycling by $\beta(t,x)$, a nonnegative, smooth function. Assume that the per unit cost of recycling increases as the amount of recycling increases (time held constant) so that $\beta_x(t,x) \geq 0$. The idea here is that it is less expensive per unit to recycle bottles and newspapers than it is to recycle bottles, newspapers, and refrigerators. Assume also that the per unit cost of recycling is nonincreasing over time (with the amount of recycling held constant) so that $\beta_t(t,x) \leq 0$. This is a standard assumption in the exhaustible resources literature; cf. Dasgupta and Heal [3, p. 179]. The total cost of recycling is $\beta(t,x)x(t)$ (i.e., the product of the per unit recycling cost β and the recycling rate x). Total cost is assumed to be a convex function of x, which implies $\beta_{xx}(t,x)x(t) + 2\beta_x(t,x) \geq 0$. Finally, it is assumed that $\beta_{xt}(t,x) \leq 0$, i.e., that an increase in t will not increase $\beta_x(t,x)$.

In addition to the disposal constraint $c(t) = x(t) + z(t)$, the consumer is bound by a budget constraint. To construct this, suppose first that the representative consumer has an income $Y(t)$ that grows over time: $\dot{Y}(t)\,(= dY/dt) > 0$. Since recycling is more expensive than landfilling, it can be assumed that the "out-of-pocket" cost of landfilling is zero. Finally, the units of measurement can be chosen so that the (exogenous) price of the consumption good is set at 1 and all other costs are measured relative to the price of the consumption good.[3] Thus, the instantaneous budget constraint is $c(t) + \beta(t, x(t))x(t) \leq Y(t)$.

As is standard in economics, it is assumed that the objective of the consumer is to maximize the total discounted "utility," an abstraction representing satisfaction or happiness. More specifically, a utility function is a realization of a preference relation between various levels of consumption. The representative consumer prefers more consumption to less, and while the marginal satisfaction he or she receives from a unit of consumption declines as consumption increases, marginal satisfaction is always

[3] The budget constraint (1.4) can be derived from a set of less simplistic assumptions as follows. Suppose the price of consumption is given as p_c, the per unit cost of landfilling waste p_z, the per unit cost of recycling p_x, and nominal income $I(t)$. (These subscripts are not meant to denote partial derivatives and will only be used in this note.) In keeping with the preceding assumptions that recycling is the more expensive waste disposal alternative, assume that $p_x > p_z$. The budget constraint is thus $p_c\, c(t) + p_z\, z(t) + p_x\, x(t) \leq I(t)$. Noting that $c(t) = x(t) + z(t)$, this simplifies to $(p_c + p_z)\, c(t) + (p_x - p_z)\, x(t) \leq I(t)$, which can now be written $c(t) + \beta\, x(t) \leq Y(t)$, where $\beta = (p_x - p_z)/(p_c + p_z)$ and $Y(t) = I(t)/(p_c + p_z)$. In this paper, we generalize this budget constraint by allowing for the possibility that β is a function of both the amount of recycling and time.

The units of measurement in solid waste management are often either "gate cubic yards" or tons. The relationship between the two units is set by an industry practice of dividing gate cubic yards by 3.3 to approximate tons. Prices are measured in dollars per unit quantity. The "trick" of essentially setting the price of the consumption good at 1 is referred to by saying that the price of consumption is the numeraire price.

positive. (This "nonsatiety" assumption may not hold for some few individuals in the real world; however, it is certainly the case that municipalities as a whole satisfy the assumption.) Letting $U(c)$ be the utility of consumption, these assumptions[4] imply $U'(c) > 0$, $U''(c) < 0$.

The discounting mentioned above reflects consumer preferences for immediate consumption rather than deferred consumption. This has been a point of contention in the natural resources literature, but is included here in a form that can allow for no discounting. Let ρ be the discount rate—usually around 2% or 3% (although the primary result of the paper is unchanged if there is no discounting, i.e., if $\rho = 0$). The problem for the representative consumer is

$$\max_{c,x,z} \int_0^T U(c(t))e^{-\rho t}dt \tag{1.1}$$

subject to

$$\frac{ds}{dt} = -z(t), \quad s(0) = s_0, \tag{1.2}$$

$$c(t) = x(t) + z(t), \tag{1.3}$$

$$c(t) + \beta(t, x(t))\, x(t) \leq Y(t), \tag{1.4}$$

$$x(t),\ z(t) \geq 0. \tag{1.5}$$

In fact, the budget constraint (1.4) must be an equality. If it were a strict inequality, then some of the "excess" income could be used to increase consumption and hence increase total utility. (Subsequent discussion will always refer to (1.4) as an equality.) Because of the concavity of the utility function and linearity of constraints, it is not hard to see that an optimal solution exists (Theorem 6.2.1 of Seierstad and Sydsæter [16, p. 358]) and is continuous (Theorem 6.1 of Fleming and Rishel [5, p. 75]).

The Hamiltonian for this problem is

$$H = U(c(t))\, e^{-\rho t} - \lambda z(t)$$

and, incorporating the two constraints, the Lagrangian is

$$L = H - \mu_1[c(t) - x(t) - z(t)] - \mu_2[c(t) + \beta(t, x(t))x(t) - Y(t)].$$

Pontryagin's maximum principle (cf. Seierstad and Sydsæter [16, p. 275]) says that at an optimal solution, it is necessary that the Hamiltonian–Lagrangian, considered as a function of the controls c, x, and z, be maximized and that the derivative of the Lagrangian is zero on intervals where the controls are continuous. So, necessary conditions are

$$L_c = U'(c(t))\, e^{-\rho t} - \mu_1 - \mu_2 = 0, \tag{1.6}$$

[4]It is also assumed that the utility function is such that $c(t) > 0$ for all t because if consumption is zero, then both recycling and landfilling must be zero and the optimization problem becomes uninteresting. One way to insure that $c(t) > 0$ is to place hypotheses on the utility function. A common hypothesis is that $\lim_{c \to 0^+} U'(c) = +\infty$. Utility functions such as $U(c) = \log c$ or $U(c) = c^\alpha$, $0 < \alpha < 1$, guarantee that there is some consumption for all $t > 0$.

$$L_x = \mu_1 - \mu_2\,[\beta_x(t,x(t))\,x(t) + \beta(t,x(t))] \le 0, \quad x(t) \ge 0, \tag{1.7}$$

$$L_z = \mu_1 - \lambda \le 0, \quad z(t) \ge 0, \tag{1.8}$$

$$\dot{\lambda} = 0. \tag{1.9}$$

The last condition follows from the requirement that $\dot{\lambda} = -\partial H/\partial s$ and the fact that H is not explicitly dependent on the state variable s. In fact, λ is nonnegative in this problem.

Since it is required that $c(t)$, $x(t)$, and $z(t)$ all be nonnegative, there exists an expression that serves as an upper bound for $x(t)$. To be more explicit, note that at any time t, the relationship $c(t) = x(t) + z(t)$ implies that the maximum amount of recycling occurs when there is no landfilling (i.e., $z(t) = 0$). But if $z(t) = 0$, then $c(t) = x(t)$ and the budget constraint implies $x(t) + \beta(t,x(t))x(t) = Y(t)$. It can be checked (using a form of the implicit function theorem, if desired) that the resulting equation $Y(t) = (1 + \beta(t,x(t)))x(t)$ gives x implicitly as a function of t, and this function describes the amount of recycling when there is no landfilling. Thus, for all t, $x(t)$ is constrained to lie in the ("implicit") interval $0 \le x(t) \le Y(t)/(1+\beta(t,x(t)))$.

The implications of these remarks will be developed more fully shortly, but the general shape of the functions can be summarized as follows. In general, the planning horizon decomposes into three intervals. On the first interval, $0 < t < t_1$, the Hamiltonian is maximized with an endpoint solution, $x(t) = 0$: the municipality is doing no recycling and all waste is going to the landfill; on the second interval, $t_1 < t < t_2$, the necessary conditions (1.6)–(1.9) are operative and the municipality is both recycling and landfilling; finally, for $t_2 < t < T$, an endpoint solution is also obtained: the landfill is exhausted so that $z(t) = 0$, and recycling occurs at its maximal level described above as an implicit function.

2. Results. This section describes the properties of the solution that can be gleaned from the necessary conditions (1.6)–(1.9). It is first shown that these conditions imply that recycling is growing, $\dot{x}(t)(= dx/dt) > 0$, during an interval of time when the municipality is simultaneously landfilling and recycling. This in turn allows us to describe the general shape of the $x(t)$ curve. The section concludes with a comparative dynamic analysis of the effect of different sizes of landfills, s_0.

2.1. The Recycling Path, $x(t)$. Suppose that there is an interval $t_1 < t < t_2$ during which the municipality is both landfilling and recycling. For such t, the necessary conditions (1.6)–(1.9) hold with $0 < x(t) < Y(t)/(1 + \beta(t,x(t)))$. Then $z(t) > 0$ as well, so both recycling and landfilling are occurring. The necessary conditions in this case imply $L_x = 0$ and $L_z = 0$. The latter equation implies that $\mu_1 = \lambda$, and so $L_x = 0$ implies $\mu_2 = \lambda\,(\beta_x(t,x(t))x(t) + \beta(t,x(t)))^{-1}$. Using these representations in the remaining condition $L_c = 0$ gives

$$U'(c(t))\,e^{-\rho t} = \lambda\left(1 + \frac{1}{\beta_x(t,x(t))x(t) + \beta(t,x(t))}\right). \tag{2.1}$$

This equation will allow us to determine the sign of $\dot{x}(t)$, as shown next.

Multiply both sides of (2.1) by $e^{\rho t}$, take the time derivative of both sides (and suppress the time arguments of functions) to get

$$U''(c)\dot{c} = \lambda\,e^{\rho t}\left(\rho\left(1 + \frac{1}{\beta_x x + \beta}\right) - \frac{(\beta_{xx}x + 2\beta_x)\dot{x} + (\beta_{xt}x + \beta_t)}{(\beta_x x + \beta)^2}\right). \tag{2.2}$$

Use the budget constraint $c(t) = Y(t) - \beta(t, x(t))x(t)$ to calculate $\dot{c}$: $\dot{c} = \dot{Y} - (\beta_x \dot{x} x + \beta_t x + \beta \dot{x})$. Substitute this expression into the left-hand side of (2.2) to get

$$(2.3) \quad U''(c)(\dot{Y} - (\beta_x \dot{x} x + \beta_t x + \beta \dot{x})) = U''(c)(\dot{Y} - \beta_t x) - U''(c)(\beta_x x + \beta)\dot{x}.$$

Rewrite the right-hand side of (2.2) as an affine function of $\dot{x}$:

$$(2.4) \quad \lambda e^{\rho t} \left(\rho \left(1 + \frac{1}{\beta_x x + \beta} \right) - \frac{(\beta_{xt} x + \beta_t)}{(\beta_x x + \beta)^2} \right) - \lambda e^{\rho t} \left(\frac{\beta_{xx} x + 2\beta_x}{(\beta_x x + \beta)^2} \right) \dot{x}.$$

Equate the right-hand side of (2.3) and expression (2.4) and solve for $\dot{x}$ to get

$$\dot{x}(t) = \frac{U''(c)(\dot{Y} - \beta_t x) - \lambda e^{\rho t}[\rho(1 + \frac{1}{\beta_x x + \beta}) - \frac{\beta_{xt} x + \beta_t}{(\beta_x x + \beta)^2}]}{U''(c)(\beta_x x + \beta) - \lambda e^{\rho t}(\beta_x x + \beta)^{-2}(\beta_{xx} x + 2\beta_x)}.$$

Recall that by assumption, $U'' < 0$, $\beta_t \le 0$, $\beta_{xt} \le 0$, and $\beta_{xx} x + 2\beta_x \ge 0$. All other quantities are positive, including λ. (It can be shown that $\lambda = 0$ only when $x(t) = 0$ for all t.) It follows that both the numerator and the denominator are negative, making $\dot{x}(t) > 0$. So on any interval during which both landfilling and recycling are occurring, the rate of recycling is increasing. This is the primary result promised in the introduction.

The preceding information on the sign of $\dot{x}$ is enough to allow us to construct a typical recycling function on $[0, T]$. Recall first that $x(t)$ is continuous. Let $\mathcal{T} = \{t \in [0, T] : 0 < x(t) < Y(t)/(1 + \beta(t, x(t)))\}$ and let $t_1 = \inf \mathcal{T}$, while $t_2 = \sup \mathcal{T}$. Since $\dot{x}(t) > 0$ on $\mathcal{T}$, $t_1 < t_2$. For the purposes of the discussion here, we will assume $0 < t_1 < t_2 < T$ and explain later what may occur if this condition does not hold. By continuity, $x(t_1) = 0$ and so $x(t) = 0$ for $t < t_1$. That is, the $x(t)$ part of the solution to conditions (1.6)–(1.9) is the zero endpoint of the interval of admissible values for $x(t)$. For $t > t_2$, again by continuity, $x(t)$ must be at the maximum allowable value. So for $t > t_2$, $x(t)$ is the function given implicitly by $Y(t) = (1 + \beta(t, x(t)))x(t)$ and $z(t_2) = 0$. It is easy to check that $x(t)$ given by the preceding equation is increasing. Indeed, taking the time derivative of both sides gives

$$\dot{Y}(t) = (1 + \beta(t, x(t)))\dot{x}(t) + (\beta_t(t, x(t)) + \beta_x(t, x(t))\dot{x}(t))x(t).$$

Solve for $\dot{x}$ to get

$$\dot{x}(t) = \frac{\dot{Y}(t) - \beta_t(t, x(t))x(t)}{1 + \beta(t, x(t)) + \beta_x(t, x(t))x(t)} > 0.$$

A description of the optimal recycling path, $x(t)$, is now possible. In general the recycling path is piecewise defined, with three pieces.[5] For the period of time before t_1, all waste is landfilled and thus recycling is zero. At t_1, recycling begins and it increases for the rest of the planning period. At time t_2 the landfill is exhausted and so all waste is disposed of by recycling.

2.2. The Comparative Dynamics of Landfill Capacity, s_0. A key parameter for a city is the initial size of its landfill, s_0. The objective of comparative dynamics is to determine the effect of s_0 on consumption, recycling, landfilling, and the "user cost."

[5]It should be noted that depending on the functional forms and parameter values, a particular recycling path might not have all three pieces of a solution; a municipality might, for example, have sufficient landfill capacity that recycling never begins.

This is accomplished by investigating the derivative of these items with respect to s_0. Begin by writing the landfill capacity constraint as

$$s_0 = \int_0^T z(t)\,dt = \int_0^{t_1} Y(t)\,dt + \int_{t_1}^{t_2} z(t)\,dt, \tag{2.5}$$

recalling that $c(t) = z(t) = Y(t)$ for $t < t_1$. Since t_1, t_2, and λ all depend (implicitly) on the parameter s_0, we differentiate (2.5) with respect to s_0:

$$1 = Y(t_1)\frac{dt_1}{ds_0} + z(t_2)\frac{dt_2}{ds_0} - z(t_1)\frac{dt_1}{ds_0} + \int_{t_1}^{t_2} \frac{\partial z(t)}{\partial s_0}\,dt. \tag{2.6}$$

To determine the sign of $\partial z(t)/\partial s_0$ we begin by showing that the first three terms on the right side of (2.6) yield zero. Since $x(t) = 0$ for $0 < t < t_1$, we have $z(t) = Y(t)$ on the interval $(0, t_1)$ from (1.3)–(1.4). Recall that the optimal solutions are continuous as functions of t, so continuity of $z(t)$ implies $z(t_1) = Y(t_1)$. Since the landfill is exhausted at t_2, all landfilling stops and continuity implies $z(t_2) = 0$. We refer to the two conditions $z(t_1) = Y(t_1)$ and $z(t_2) = 0$ as "continuity conditions." As will be seen in the next section, these two equations together with the landfill capacity constraint will enable the computation of t_1, t_2, and the adjoint variable λ and thus complete the construction of the solutions $c(t)$, $x(t)$, and $z(t)$. For now, it follows from the continuity conditions that the sum of the first three terms on the right side of (2.6) is zero, leaving $1 = \int_{t_1}^{t_2} \partial z(t)/\partial s_0 dt$. Thus, $\partial z(t)/\partial s_0$ must be positive on some time interval inside (t_1, t_2). To determine the signs of the derivatives of the other controls as functions of initial landfill capacity, use (1.3)–(1.4) to eliminate $c(t)$ and write $z(t)$ as a function of $x(t)$: $z(t) = Y(t) - (1 + \beta(t, x(t)))x(t)$. Differentiate with respect to s_0 to get

$$\frac{\partial z(t)}{\partial s_0} = -(1 + \beta(t, x(t)) + \beta_x(t, x(t))x(t))\frac{\partial x(t)}{\partial s_0}, \tag{2.7}$$

and from constraint (1.4),

$$\frac{\partial c(t)}{\partial s_0} = -(\beta(t, x(t)) + \beta_x(t, x(t))x(t))\frac{\partial x(t)}{\partial s_0}. \tag{2.8}$$

These two conditions imply that for the time interval during which $\partial z(t)/\partial s_0 > 0$ it is also the case that $\partial x(t)/\partial s_0 < 0$ and $\partial c(t)/\partial s_0 > 0$.

As discussed in the next paragraph, it is also of interest to determine the effect of landfill size on the adjoint variable λ. Differentiate (with respect to s_0) the necessary condition as expressed in (2.1) (and suppress the arguments of functions in the notation):

$$U'' e^{-\rho t}\frac{\partial c}{\partial s_0} + \lambda\frac{\beta_{xx}x + 2\beta_x}{(\beta_x x + \beta)^2}\,\frac{\partial x}{\partial s_0} = \left(1 + \frac{1}{\beta_x x + \beta}\right)\frac{\partial \lambda}{\partial s_0}. \tag{2.9}$$

Thus, for the time interval when $\partial z(t)/\partial s_0 > 0$ (and hence $\partial x(t)/\partial s_0 < 0$ and $\partial c(t)/\partial s_0 > 0$) it follows that $\partial\lambda/\partial s_0 < 0$. But since λ does not vary with time, $\partial\lambda/\partial s_0 < 0$ for the entire planning period. To summarize,

$$\frac{\partial z(t)}{\partial s_0} > 0, \quad \frac{\partial x(t)}{\partial s_0} < 0, \quad \frac{\partial c(t)}{\partial s_0} > 0, \quad \text{and } \frac{\partial \lambda}{\partial s_0} < 0$$

for the entire interval of time between t_1 and t_2.

To interpret these results, it is helpful to define the maximized value of the objective function (1.1) (which gives the total discounted utility for the representative consumer) as an implicit function of s_0: $V(s_0)$. It can be shown that the adjoint variable λ is dV/ds_0 [16, p. 210]. Thus, λ is interpreted as the marginal value of an increase in initial landfill space. In the exhaustible resources literature λ is sometimes called the "user cost." The interpretation of $V''(s_0) = d\lambda/ds_0 < 0$ is simply that increases in initial landfill capacity s_0 (all other factors being constant) will decrease the user cost.

The only exception to this result is when (for some functional forms and sets of parameter values) λ is zero, in which case the landfill is not exhausted, the landfill capacity constraint is not binding, and no recycling is done. In the more typical cases, i.e., for municipalities for which the landfill capacity constraint is binding, any small decrease in initial capacity s_0 increases the user cost, i.e., increases the marginal value of an increment of landfill capacity. It also follows in this case that a decrease in initial capacity s_0 implies an increase in the amount recycled at any moment (except, of course, for the time period when recycling is 0). An increase in landfill capacity implies a decrease in recycling. Intuitively, the model predicts that the more landfill capacity a municipality has, the less it needs to recycle and the more landfilling it will do. Further, consumption at any fixed time between t_1 and t_2 will increase with an increase in s_0 (and will stay the same for $t \leq t_1$ and $t \geq t_2$). So it follows that an increase in landfill capacity increases total discounted utility unless, of course, the landfill capacity is so large that the landfill constraint is nonbinding.

3. Examples and Exercises. This section considers a couple of examples showing how the constraints and the necessary and endpoint conditions can be used to characterize an optimal solution. Exercises that apply and extend the analysis in various ways are also given. The examples and exercises consolidate the recycling results above and show the diversity of other aspects of an optimal waste management plan such as consumption and landfilling. Note, however, that the parameter values chosen for the examples and exercises are illustrative only and are not meant to be those of any particular city.

3.1. Example 1. Assume a utility function $U(c(t)) = \ln c(t)$. The log function is a common choice for an example of a utility function in that it is increasing and concave and some consumption is always optimal: $c(t) > 0$. Assume that income is growing exponentially: $Y(t) = Y_0 e^{\delta t}$, where δ is the growth rate of income. To illustrate optimal controls with the most general form, we assume that the landfill constraint is binding, so that the user cost λ is positive. Finally, in this first example, assume that the per unit recycling cost is a positive constant, $\beta(t, x(t)) \equiv \beta$.

While recycling is increasing for $t_1 < t < t_2$ as shown in section 2, in this example, as will be seen, consumption and landfilling decrease during this time interval. Since $U'(c(t)) = 1/c(t)$, equation (2.1), which holds between t_1 and t_2, implies $c(t) = (\beta/(\lambda(1+\beta)))e^{-\rho t}$. Recalling from (1.9) that λ is constant (and positive), this immediately implies $\dot{c}(t) < 0$ for t in (t_1, t_2). Rewrite the budget constraint (1.4) as $x(t) = (Y(t) - c(t))/\beta$ so that $x(t) = (Y_0/\beta)e^{\delta t} - (1/(\lambda(1+\beta)))e^{-\rho t}$. Use the disposal constraint (1.3) to write $z(t) = c(t) - x(t) = (1/\lambda)e^{-\rho t} - (Y_0/\beta)e^{\delta t}$. It follows that $\dot{z}(t) < 0$ in (t_1, t_2).

To completely identify the three functions $c(t)$, $x(t)$, and $z(t)$, it remains to compute t_1, t_2, and λ. Recall that t_1 is defined as the "last time" at which recycling is zero. Thus, $c(t_1) = Y(t_1) = z(t_1)$ and so (from the latter equality)

$$Y_0 e^{\delta t_1} = \frac{e^{-\rho t_1}}{\lambda} - \frac{Y_0 e^{\delta t_1}}{\beta}.$$

Similarly, t_2 is defined as the "first moment" at which landfilling is zero, i.e., $z(t_2) = 0$ and so

$$\frac{e^{-\rho t_2}}{\lambda} - \frac{Y_0 e^{\delta t_2}}{\beta} = 0.$$

It is now possible to use the stock constraint (2.5) and the forms of $Y(t)$ and $z(t)$ to calculate λ. To do this it will be useful to solve the preceding expressions for t_1 and t_2, respectively:

$$t_1 = \frac{-1}{\delta+\rho} \ln\left(\frac{\lambda Y_0(1+\beta)}{\beta}\right) \text{ and } t_2 = \frac{-1}{\delta+\rho} \ln\left(\frac{\lambda Y_0}{\beta}\right). \tag{3.1}$$

Thus,

$$\begin{aligned}
s_0 &= \int_0^T z(t)\,dt = \int_0^{t_1} Y(t)\,dt + \int_{t_1}^{t_2} z(t)\,dt \\
&= \int_0^{t_1} Y_0 e^{\delta t}\,dt + \int_{t_1}^{t_2} \frac{e^{-\rho t}}{\lambda} - \frac{Y_0 e^{\delta t}}{\beta}\,dt \\
&= \frac{Y_0}{\delta}(e^{\delta t_1} - 1) - \frac{1}{\rho\lambda}(e^{-\rho t_2} - e^{-\rho t_1}) - \frac{1}{\delta\beta} Y_0 (e^{\delta t_2} - e^{\delta t_1}) \\
&= \frac{Y_0}{\delta}\left(\left(\frac{\lambda Y_0(1+\beta)}{\beta}\right)^{\frac{-\delta}{\delta+\rho}} - 1\right) - \frac{1}{\rho\lambda}\left(\left(\frac{\lambda Y_0}{\beta}\right)^{\frac{\rho}{\delta+\rho}} - \left(\frac{\lambda Y_0(1+\beta)}{\beta}\right)^{\frac{\rho}{\delta+\rho}}\right) \\
&\quad - \frac{1}{\delta\beta} Y_0 \left(\left(\frac{\lambda Y_0}{\beta}\right)^{\frac{-\delta}{\delta+\rho}} - \left(\frac{\lambda Y_0(1+\beta)}{\beta}\right)^{\frac{-\delta}{\delta+\rho}}\right).
\end{aligned}$$

Adding Y_0/δ to both sides and writing $\frac{1}{\lambda}$ as $\lambda^{\frac{-(\delta+\rho)}{\delta+\rho}}$ in the second term, we can factor $\lambda^{\frac{-\delta}{\delta+\rho}}$ from the remaining terms:

$$\begin{aligned}
s_0 + \frac{Y_0}{\delta} &= \lambda^{\frac{-\delta}{\delta+\rho}}\left[\frac{Y_0}{\delta}\left(\frac{Y_0(1+\beta)}{\beta}\right)^{\frac{-\delta}{\delta+\rho}} - \frac{1}{\rho}\left(\frac{Y_0}{\beta}\right)^{\frac{\rho}{\delta+\rho}}\left(1 - (1+\beta)^{\frac{\rho}{\delta+\rho}}\right)\right. \\
&\qquad \left. - \frac{1}{\beta\delta} Y_0 \left(\frac{Y_0}{\beta}\right)^{\frac{-\delta}{\delta+\rho}}\left(1 - (1+\beta)^{\frac{-\delta}{\delta+\rho}}\right)\right] \\
&= \lambda^{\frac{-\delta}{\delta+\rho}} Y_0^{\frac{\rho}{\delta+\rho}}\left[\frac{1}{\delta}\left(\frac{1}{\beta}\right)^{\frac{-\delta}{\delta+\rho}}(1+\beta)^{\frac{-\delta}{\delta+\rho}} - \frac{1}{\rho}\left(\frac{1}{\beta}\right)^{\frac{\rho}{\delta+\rho}}\left(1 - (1+\beta)^{\frac{\rho}{\delta+\rho}}\right)\right. \\
&\qquad \left. - \frac{1}{\delta}\left(\frac{1}{\beta}\right)^{\frac{\rho}{\delta+\rho}}\left(1 - (1+\beta)^{\frac{-\delta}{\delta+\rho}}\right)\right].
\end{aligned}$$

From this expression, it is clear that λ can be written as a function of the other parameters of the model. Some tedious algebraic manipulations give

$$\lambda = \left(\frac{s_0}{Y_0} + \frac{1}{\delta}\right)^{-\frac{\delta+\rho}{\delta}} \frac{\beta}{Y_0}\left[\frac{1}{\beta}\left(\frac{1}{\delta} + \frac{1}{\rho}\right)\left((1+\beta)^{\frac{\rho}{\delta+\rho}} - 1\right)\right]^{\frac{\delta+\rho}{\delta}}. \tag{3.2}$$

The complete optimal paths for this example have now been described. To summarize, the optimal paths are, for consumption:

$$c(t) = \begin{cases} Y_0 e^{\delta t}, & 0 \leq t < t_1, \\ \frac{\beta}{1+\beta}\frac{e^{-\rho t}}{\lambda}, & t_1 \leq t \leq t_2, \\ \frac{Y_0 e^{\delta t}}{1+\beta}, & t_2 < t \leq T; \end{cases}$$

for landfilling:

$$z(t) = \begin{cases} Y_0 e^{\delta t}, & 0 \leq t < t_1, \\ \frac{e^{-\rho t}}{\lambda} - \frac{Y_0 e^{\delta t}}{\beta}, & t_1 \leq t \leq t_2, \\ 0, & t_2 < t < T; \end{cases}$$

and for recycling:

$$x(t) = \begin{cases} 0, & 0 \leq t < t_1, \\ \frac{Y_0 e^{\delta t}}{\beta} - \frac{1}{1+\beta}\frac{e^{-\rho t}}{\lambda}, & t_1 \leq t \leq t_2, \\ \frac{Y_0 e^{\delta t}}{1+\beta}, & t_2 < t \leq T, \end{cases}$$

where λ, t_1, and t_2 are found from (3.1) and (3.2) above.

These paths are shown in Figure 3.1, where time is on the horizontal axis and consumption, landfilling, and recycling, respectively, are on the vertical axes. Along an optimal path, beginning with $t = 0$, consumption rises with income (and is equal to income) until the switch time t_1. For this initial interval, $0 < t < t_1$, all waste is landfilled, so the landfilling path is the same as the consumption path, and recycling is zero. In the middle period, $t_1 < t < t_2$, there is a mixture of landfilling and recycling, the former being phased out while the latter is phased in as the landfill reaches its capacity. The solution is given by the necessary conditions (1.6)–(1.9). After the second switch time t_2, consumption follows the path $Y_0 e^{\delta t}/(1+\beta)$, there is no landfilling, and all waste is recycled. Notice that the path $Y_0 e^{\delta t}/(1+\beta)$ is below the income path because the backstop method of waste disposal is more expensive than landfilling, and thus less income remains to be spent on consumption. The municipality starts out by consuming along the upper path $Y_0 e^{\delta t}$ but must eventually shift to the lower path $Y_0 e^{\delta t}/(1+\beta)$ (assuming the landfill space is exhausted). Equation (2.1) determines the optimal way to make this shift, i.e., the optimal transition path between the two curves $Y_0 e^{\delta t}$ and $Y_0 e^{\delta t}/(1+\beta)$. For the functional forms of Example 1, as shown in Figure 3.1, it is optimal for consumption to actually fall between times t_1 and t_2.

Figure 3.1 illustrates the optimal path for this specific example. But it may be worth noting which of the characteristics of an optimal path depend on functional forms and which do not. The most important characteristic that is dependent on functional forms of utility, income, and cost (and parameter values) is that not all segments of paths as illustrated may actually occur. For example, the value of t_1 as calculated from (3.1) and (3.2) may be less than zero. In such a case t_1 is set to zero and the municipality does not have an initial period during which landfilling is used exclusively as the method of waste disposal. If the value of t_1 as calculated from (3.1) and (3.2) is greater than T, then t_1 is interpreted to be equal to T. In such a case an optimal solution is to never introduce recycling, since the initial landfill capacity is large enough to contain all waste for the entire planning period. Another result that depends on the functional forms is that, as in this particular example, consumption

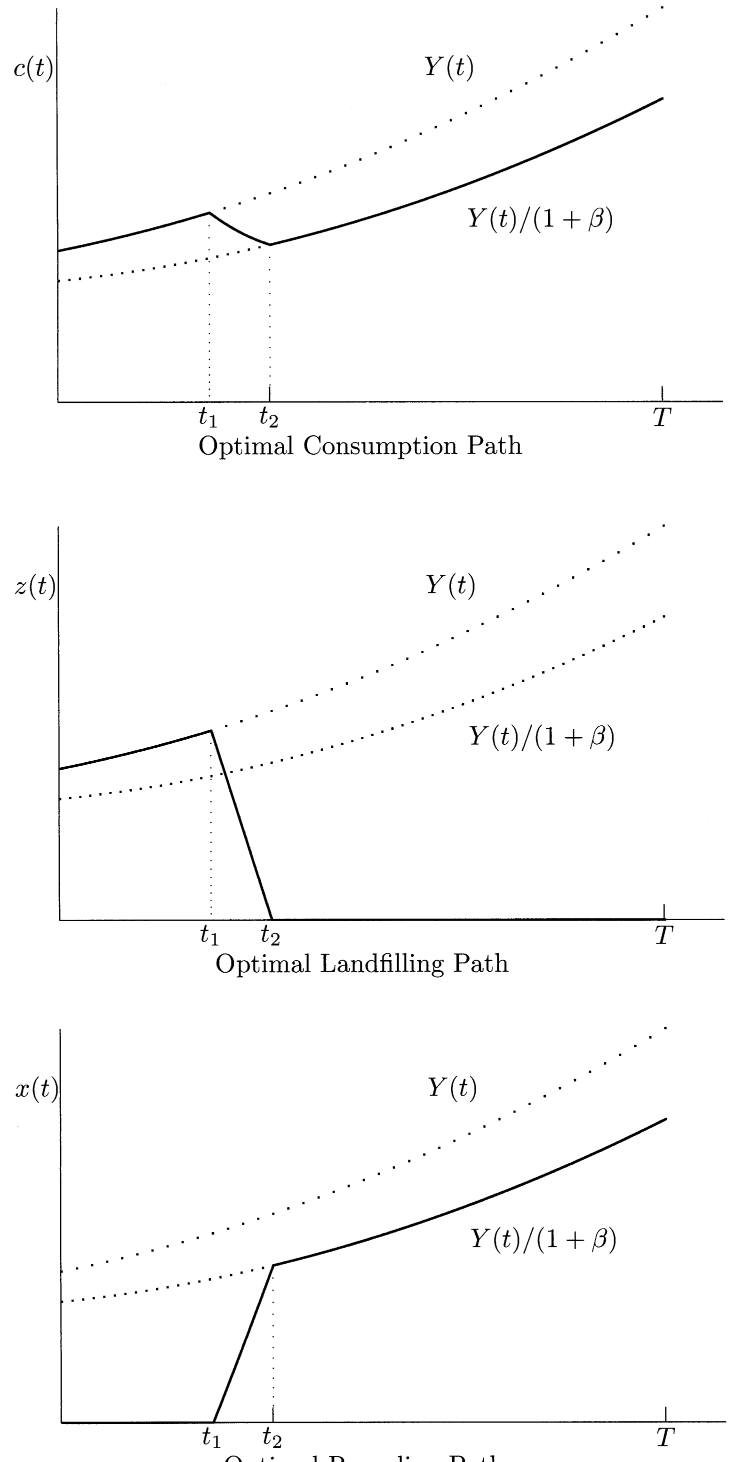

Fig. 3.1 *Optimal controls in the case $\beta(t, x)$ is constant.*

and landfilling always fall between times t_1 and t_2. Other utility functions, for example, can imply a different path for the transition interval between the "upper" and "lower" consumption paths. It will be seen in Example 2 that if the per unit marginal cost depends linearly on t and x, then consumption can indeed rise in the interval $t_1 < t < t_2$.

Results that do not depend on functional forms include the fact that recycling always increases once it begins and that consumption grows (because income is growing) when either landfilling or recycling is zero. And, for solutions that involve more than one of the segments shown in the figure, the pieces of the solution will always occur in the order shown. That is, an interval without recycling occurs before an interval during which both landfilling and recycling occur, which in turn occurs before an interval without landfilling.

For the exercises that follow, we choose, somewhat arbitrarily, various sets of parameters. In general, s_0 and Y_0 are large relative to the other parameters. Parameters δ and ρ are small, say, around .03 (i.e., 3% rates); recall that δ is the growth rate of income and that ρ, the discount rate, can actually be zero. Since waste disposal costs should be a small fraction of the price of consumption, the cost parameter β should be significantly less than 1, which is the implicit price of a unit of consumption.

Exercise 1. Let $\beta = 0.08$, $T = 10$, $s_0 = 50$, $Y_0 = 25, \rho = 0.04,$ and $\delta = 0.03$. Calculate the user cost, λ, for this set of parameters. What happens to the user cost if the stock size $s_0 = 40$? Calculate $t_2 - t_1$ for both stock sizes. Show that in general the time during which recycling is phased in, $t_2 - t_1$, is independent of s_0 and Y_0.

Exercise 2. (a) Experiment with different values of the initial landfill capacity s_0 to find a solution with a value of t_1 computed from (3.1) and (3.2) that is less than zero. For such a parameter set, the municipality will begin by using both landfilling and recycling to dispose of its waste. Construct the optimal solution for this set of parameters.

Repeat the exercise but adjust the initial income Y_0 rather than landfill capacity to find "$t_1 < 0$." Repeat again, this time adjusting the marginal recycling cost β to find another instance for which t_1 is initially computed to be less than zero.

(b) Repeat part (a) with the goal of finding parameter sets with computed values of $t_2 > T$. A municipality with such a parameter set will never have an interval of time (during the planning horizon) for which some landfilling is not optimal. Graph the solutions $c(t)$, $x(t)$, and $z(t)$.

(c) What is the interpretation if computed values of t_1 and t_2 are both less than zero? both greater than the planning horizon T?

Exercise 3. Many states in the United States require municipalities to recycle some proportion of their waste (typically between 25% and 50%) or to develop recycling plans to accomplish this goal. Calculate the initial landfill capacity s_0, where the municipality will recycle 25% by the end of the planning period. What are the effects of these laws on municipalities with large landfills? Are they necessary? (The latter two questions, of course, introduce issues of political economy into the discussion.)

3.2. Example 2. The constant marginal cost function of the preceding example may be too simplistic an assumption. In this example it will be assumed that the cost function is an affine function of both the rate of recycling and time. One result of this more general cost function is that the values of t_1, t_2, and λ can be computed only as implicit functions of the parameters. In keeping with the general assumptions in Dasgupta and Heal [3] concerning recycling costs, assume that costs rise with the level of recycling but fall when considered (only) as a function of time. Thus assume

$\beta(t,x) = a_1x - a_2t + a_3$, where $a_1, a_2, a_3 > 0$. The assumptions of a log utility function $U(c(t)) = \ln c(t)$, exponential income growth $Y(t) = Y_0e^{\delta t}$, and positive user cost ($\lambda > 0$) are retained from the preceding example.

Assume there is an interval $t_1 < t < t_2$ during which the necessary conditions (1.6)–(1.9) are active. Consumption can now be written only by referring to recycling. Use (2.1) to get

$$\begin{aligned} c(t) &= \frac{1}{\lambda}\left(1 + \frac{1}{\beta_x(t,x(t))x(t) + \beta(t,x(t))}\right)^{-1} e^{-\rho t} \\ &= \frac{1}{\lambda}\left(\frac{2a_1x(t) - a_2t + a_3}{2a_1x(t) - a_2t + a_3 + 1}\right) e^{-\rho t}. \end{aligned}$$

The budget constraint (1.4), $Y(t) = c(t) + \beta(t,x(t))x(t)$, can be combined with the preceding equation to write recycling $x(t)$ as an implicit function of time (for $t_1 < t < t_2$):

$$Y(t) - \beta(t,x(t))x(t) = \frac{1}{\lambda}\left(\frac{2a_1x(t) - a_2t + a_3}{2a_1x(t) - a_2t + a_3 + 1}\right) e^{-\rho t}. \tag{3.3}$$

Landfilling during the same interval is again given only as a function of $x(t)$, by using the disposal constraint (1.3), $c(t) = x(t) + z(t)$:

$$z(t) = \frac{1}{\lambda}\left(\frac{2a_1x(t) - a_2t + a_3}{2a_1x(t) - a_2t + a_3 + 1}\right) e^{-\rho t} - x(t). \tag{3.4}$$

As in the preceding example, it remains to use the continuity conditions ($z(t_1) = Y(t_1)$, $z(t_2) = 0$) and the stock constraint (2.5) to find t_1, t_2, and λ. Since it is not possible to solve for t_1, t_2, and λ as explicit functions of the other parameters, one way to approximate these values is to do so iteratively. Namely, make an initial guess of λ (based perhaps on the easier case in Example 1), compute t_1, t_2, and landfill size s_0, and then adjust the value of λ to get the desired value of s_0. (We have found *Mathematica* to be useful in this regard.) Using this procedure, to find t_1, note that $x(t_1) = 0$, so solve (numerically) equation (3.3) for t_1. Next observe that for $t \geq t_2$, $c(t_2) = x(t_2)$ (since the landfill is exhausted), so the budget constraint becomes $Y(t) = (1 + \beta(t,x(t)))\,x(t)$. To find t_2 and $x(t_2)$, use this version of the budget constraint and (3.3), both evaluated at t_2, and solve numerically for t_2 and $x(t_2)$. To compute the landfill capacity (recall that we are taking λ as given, for the moment), use (3.3) to find values of t for $t_1 < t < t_2$ and then compute values for landfilling using (3.4). Compute enough values to be able to numerically integrate the stock constraint (2.5): $s_0 = \int_0^T z(t)\,dt = \int_0^{t_1} Y(t)dt + \int_{t_1}^{t_2} z(t)dt$. The user cost λ can now be adjusted to achieve the desired value of s_0 after a few iterations of the process.

General results when assuming an affine cost function $\beta(t,x(t))$ are more rare than in the previous example. Of course, it is still the case that recycling, once initiated, will continue at an increasing rate for the duration of the planning period. Since this cost function generalizes the constant function from Example 1, it follows that there are parameter values $a_i \neq 0$ ($i = 1,2,3$) for which consumption $c(t)$ falls during the phase in period $t_1 < t < t_2$. However, there also exist values of the a_i for which consumption grows for $t_1 < t < t_2$ and still other values which yield a nonmonotone consumption path for $t_1 < t < t_2$. Similarly, while most often landfilling will fall for

$t_1 < t < t_2$, there are parameter values for which landfilling will grow briefly during this period. The implication of this variability of the consumption and landfilling paths seems to be that some aspects of optimal waste management systems differ considerably between communities.

Exercise 4. Starting with the parameter values given in Exercise 1, and letting $a_1 = 0.001$, $a_2 = 0.001$, and $a_3 = 0.08$, graph the optimal $c(t)$, $x(t)$, and $z(t)$ and verify that $c(t)$ is decreasing during (t_1, t_2). Next increase a_1 and a_2 to get a consumption function that increases for all t.

Exercise 5. There is anecdotal evidence suggesting that some people perceive utility from recycling that is distinct from the utility of consumption [12, p. 4]. To allow for this possibility, reformulate the optimal control problem as follows. Arguments can be made for either a concave or a convex utility of recycling, so compromise with a linear function, $kx(t)$, with k small in comparison to the utility of consumption. The problem is now to maximize $\int_0^T (U(c(t))+kx(t))dt$ for some constant k, subject to the other constraints given in section 1. Using log utility, exponential income growth, and (for an easier exercise) a constant marginal cost of recycling, is it still the case that recycling, once begun, will grow over time? Is consumption monotone?

4. Conclusion. The focus of this paper is on the adoption of recycling as a "backstop" waste disposal method for municipalities with a fixed landfill capacity. The model predicts that all municipalities will introduce recycling by the end of the planning period unless their initial landfill is so large that it can contain all the waste generated during the planning period. Once begun, recycling is phased in over time. After a municipality begins recycling, it will continue to recycle. Further, the amount of recycling increases because income rises over time. The amount of recycling depends, other things being equal, on the initial landfill capacity. The smaller the initial landfill capacity the greater the amount of recycling. By the end of the planning period, municipalities with a small enough landfill capacity may very well be recycling all of their waste.

In the current political environment, many state governments are trying to encourage municipalities to recycle, often by requiring that a certain proportion of waste be recycled or that a recycling plan be in place by a certain date. While these efforts may contribute to the greater good, such regulations need to be written in a way that is sensitive to the differences between municipalities (i.e., initial landfill capacities) that lead to differences in their optimal waste management plans.

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